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Derivation of an alternative expression for process noise in MKS

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Problem statement:

In early spring 2004, Sweung Cheung derived a new expression for computing process noise Q in the standard Multiscale Kalman Smoother MKS algorithm.

Derivation of an alternative expression for $Q(s)$

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By Sweungwon Cheung

Forward (Down-sweep) Model

Let the spatial interval be $0 \le s \le M$ and spatial state model is

$$\begin{aligned} x(s) &= \Phi(s)x(Bs) + \Gamma(s)w(s) \\ y(s) &= H(s)x(s) + v(s) \end{aligned} \qquad (1)$$

where w(s) is white process and uncorrelated x(0)

$$E[w(s)] = 0 \quad E[w(s)w^T(t)] = I\delta(s-t)$$
$$E[x(0)] = 0 \quad E[x(0)x^T(0)] = P_x(0)$$

and forward orthogonality is satisfied, i.e. $E[x(0)w^T(s)] = 0$ for $s \ge 0$

Backward (Up-sweep) Model

The backward model can be get from (1) by just reversing the direction of spatial.

$$\begin{aligned} x(Bs) &= \Phi^{-1}(s)x(s) - \Phi^{-1}(s)\Gamma(s)w(s) \\ y(s) &= H(s)x(s) + v(s) \end{aligned} \qquad (2)$$

The backward state process is still Markove. But it does not satisfy the backward orthogonality between x(M) and w(s) for $s \le M$.

We can define w(s) as following by Markovity

$$\begin{aligned} w(s) &= E[w(s) \mid x(s), x(s+1), \ldots . x(M)] + \tilde{w}(s) \\ &= E[w(s) \mid x(s)] + \tilde{w}(s) \end{aligned}$$

where E[w(s)|x(s)] is MMSE estimate

Since $\tilde{w}(s) = w(s) - E[w(s) \mid x(s)]$ so that

$$\tilde{w}(s) \perp x(s)$$

We assumed that w(s) and x(s) are zero mean Gaussian so that we can find the following equation ([3] p324)

$$E[w(s) \mid x(s)] = E[w(s)x^T(s)]E[x(s)x^T(s)]^{-1}x(s) \qquad (3)$$

Since $x(s) = \Phi(s)x(Bs) + \Gamma(s)w(s)$ and we can substitute x(s) into (3) so that

$$E[w(s) \mid x(s)] = \Gamma^T(s)P_x^{-1}(s)x(s)$$

The backward makove model can be rewritten [1]

$$\begin{aligned} x(Bs) &= \Phi^{-1}(s)x(s) - \Phi^{-1}(s)\Gamma(s)\left(E[w(s) \mid x(s)] + \tilde{w}(s)\right) \\ &= \Phi^{-1}(s)x(s) - \Phi^{-1}(s)\Gamma(s)\Gamma^T(s)P_x^{-1}(s)x(s) - \Phi^{-1}(s)\Gamma(s)\tilde{w}(s) \\ &= \Phi^{-1}(s)\left(I - \Gamma(s)\Gamma^T(s)P_x^{-1}(s)\right)x(s) - \Phi^{-1}(s)\Gamma(s)\tilde{w}(s) \end{aligned}$$

Let

$$F(s) = \Phi^{-1}(s)\left(I - \Gamma(s)\Gamma^T(s)P_x^{-1}(s)\right)$$

$$\overline{w}(s) = -\Phi^{-1}(s)\Gamma(s)\tilde{w}(s)$$

As a result, the backward markov model is

$$\begin{aligned} x(Bs) &= F(s)x(s) + \overline{w}(s) \\ y(s) &= H(s)x(s) + v(s) \end{aligned}$$

Since

$$\begin{aligned} &P_x(s) = \Phi(s)P_x(Bs)\Phi^T(s) + \Gamma(s)I\Gamma^T(s) \\ &\Phi^{-1}(s)P_x(s) = \Phi^{-1}(s)\Phi(s)P_x(Bs)\Phi^T(s) + \Phi^{-1}(s)\Gamma(s)\Gamma^T(s) \\ &\Phi^{-1}(s) = \Phi^{-1}(s)\Phi(s)P_x(Bs)\Phi^T(s)P_x{}^{-1}(s) + \Phi^{-1}(s)\Gamma(s)\Gamma^T(s)P_x{}^{-1}(s) \\ &\Phi^{-1}(s) - \Phi^{-1}(s)\Gamma(s)\Gamma^T(s)P_x{}^{-1}(s) = P_x(Bs)\Phi^T(s)P_x{}^{-1}(s) \end{aligned}$$

so that

$$F(s) = P_x(Bs)\Phi^T(s)P_x^{-1}(s)$$

Let $Q(s) = E[\overline{w}(s)\overline{w}^T(s)]$ and

$$
\begin{aligned}
E[\tilde{w}(s)\tilde{w}^T(s)] &= E[\left(w(s) - E[w(s) \mid x(s)]\right)\left(w(s) - E[w(s) \mid x(s)]\right)^T] \\
&= E[\left(w(s) - \Gamma^T(s)P_x^{-1}(s)x(s)\right)\left(w(s) - \Gamma^T(s)P_x^{-1}(s)x(s)\right)^T] \\
&= I - \Gamma^T(s)P_x^{-1}(s)E[x(s)x(s)^T]P_x^{-T}(s)\Gamma(s) \\
&= I - \Gamma^T(s)P_x^{-1}(s)P_x(s)P_x^{-T}(s)\Gamma(s) \\
&= I - \Gamma^T(s)P_x^{-1}(s)\Gamma(s)
\end{aligned}
$$

So that

$$
\begin{aligned}
Q(s) &= E[\overline{w}(s)\overline{w}^T(s)] \\
&= \Phi^{-1}(s)\Gamma(s)E[\tilde{w}(s)\tilde{w}^T(s)]\Gamma^T(s)\Phi^{-T}(s) \\
&= \Phi^{-1}(s)\Gamma(s)\left[I - \Gamma^T(s)P_x^{-1}(s)\Gamma(s)\right]\Gamma^T(s)\Phi^{-T}(s) \\
&= \Phi^{-1}(s)\Gamma(s)\Gamma^T(s)\Phi^{-T}(s) - \Phi^{-1}(s)\Gamma(s)\Gamma^T(s)P_x^{-1}(s)\Gamma(s)\Gamma^T(s)\Phi^{-T}(s)
\end{aligned}
$$

Since

$$
\begin{aligned}
&P_x(s) = \Phi(s)P_x(Bs)\Phi^T(s) + \Gamma(s)\Gamma^T(s) \\
&\Phi^{-1}(s)P_x(t) = \Phi^{-1}(t)\Phi(s)P_x(Bs)\Phi^T(s) + \Phi^{-1}(s)\Gamma(s)\Gamma^T(s) \\
&\Phi^{-1}(s) = \Phi^{-1}(s)\Phi(s)P_x(Bs)\Phi^T(s)P_x{}^{-1}(s) + \Phi^{-1}(s)\Gamma(s)\Gamma^T(s)P_x{}^{-1}(s) \\
&\Phi^{-1}(s)\Gamma(s)\Gamma^T(s)\Phi^{-T}(s) = \Phi^{-1}(s)\Phi(s)P_x(Bs)\Phi^T(s)P_x{}^{-1}(s)\Gamma(s)\Gamma^T(s)\Phi^{-T}(s) \\
&\qquad + \Phi^{-1}(s)\Gamma(s)\Gamma^T(s)P_x{}^{-1}(s)\Gamma(s)\Gamma^T(s)\Phi^{-T}(s) \\
&\Phi^{-1}(s)\Gamma(s)\Gamma^T(s)P_x{}^{-1}(s)\Gamma(s)\Gamma^T(s)\Phi^{-T}(s) = \Phi^{-1}(s)\Gamma(s)\Gamma^T(s)\Phi^{-T}(s) + P_x(Bs)\Phi^T(s)P_x{}^{-1}(s)\Gamma(s)\Gamma^T(s)\Phi^{-T}(s)
\end{aligned}
$$

Q(s) can be rewritten

$$
Q(s) = P_x(Bs)\Phi^T(s)P_x{}^{-1}(s)\Gamma(s)\Gamma^T(s)\Phi^{-T}(s) \qquad (*)
$$

Since

$$
F(s) = P_x(Bs)\Phi^T(s)P_x^{-1}(s)
$$

and

$$
\begin{aligned}
&P_x(s) = \Phi(s)P_x(Bs)\Phi^T(s) + \Gamma(s)\Gamma^T(s) \\
&P_x(s)\Phi^{-T}(s) = \Phi(s)P_x(Bs)\Phi^T(s)\Phi^{-T}(s) + \Gamma(s)\Gamma^T(s)\Phi^{-T}(s) \\
&\Gamma(s)\Gamma^T(s)\Phi^{-T}(s) = P_x(t)\Phi^{-T}(s) - \Phi(s)P_x(Bs)
\end{aligned}
$$

so that

$$
\begin{aligned}
Q(s) &= F(s)\left(P_x(s)\Phi^{-T}(s) - \Phi(s)P_x(Bs)\right) \\
&= F(s)P_x(s)\Phi^{-T}(s) - F(s)\Phi(s)P_x(Bs)
\end{aligned}
$$

Since

$$
F(s) = P_x(Bs)\Phi^T(s)P_x^{-1}(s) \rightarrow F(s)P_x(s)\Phi^{-T}(s) = P_x(Bs)
$$

so that

$$
\begin{aligned}
Q(s) &= P_x(Bs) - F(s)\Phi(s)P_x(Bs) \\
&= \left(I - F(s)\Phi(s)\right)P_x(Bs)
\end{aligned}
$$

The equation of Q(s) which was written in Dr. slatton's handout can be derive from (*)

$$
\begin{aligned}
Q(s) &= P_x(Bs)\Phi^T(s)P_x^{-1}(s)\Gamma(s)\Gamma^T(s)\Phi^{-T}(s) \\
&= P_x(Bs)\Phi^T(s)P_x^{-1}(s)\left[P_x(s)\Phi^{-T}(s) - \Phi\ \ (s)P_x(Bs)\right] \\
&= \left[P_x(Bs)\Phi^T(s)P_x^{-1}(s)P_x(s)\Phi^{-T}(s) - P_x(Bs)\Phi^T(s)P_x^{-1}(s)\Phi\ \ (s)P_x(Bs)\right] \\
&= \left[P_x(Bs) - P_x(Bs)\Phi^T(s)P_x^{-1}(s)\Phi\ \ (s)P_x(Bs)\right] \\
&= P_x(Bs)\left[I - \Phi^T(s)P_x^{-1}(s)\Phi\ \ (s)P_x(Bs)\right]
\end{aligned}
$$

As a result, we can get two equations for Q(s)

$$
\begin{aligned}
Q(s) &= P_x(Bs)\left[I - \Phi^T(s)P_x^{-1}(s)\Phi\ \ (s)P_x(Bs)\right] \\
&= \left(I - F(s)\Phi(s)\right)P_x(Bs)
\end{aligned}
$$

In my opinion, the second equation has advantages like no inverse matrix and less computation complexity.

References

[1] G.Verghese, Thomas Kailath, "A Further Note on Backwards Markovian Models", IEEE Trans. on Info.
Theory, vol.25, pp121-124,1979

[2] K.C. Chou, Alan S. Willsky, and Albert Benveniste, "Multiscale Recursive Estimation, Data Fusion, and Regularization", IEEE Trans. on Auto. Control, vol 39, No3, 1994

[3] Steven M. Kay, "Fundamentals of Statistical Signal Processing: Vol I", Prentice Hall